GEOMETRICAL AND ANALYTICAL CHARACTERISTIC PROPERTIES OF PIECEWISE AFFINE MAPPINGS

V.V. Gorokhovik

Institute of Mathematics
The National Academy of Sciences of Belarus
Surganova st., 11, Minsk, 220072, Belarus
e-mail: gorokh@im.bas-net.by

Abstract

Let X and Y be finite dimensional normed spaces, $\mathcal{F}(X,Y)$ a collection of all mappings from X into Y. A mapping $P \in \mathcal{F}(X,Y)$ is said to be piecewise affine if there exists a finite family of convex polyhedral subsets covering X and such that the restriction of P on each subset of this family is an affine mapping. In the paper we prove a number of characterizations of piecewise affine mappings. In particular we show that a mapping $P : X \to Y$ is piecewise affine if and only if for any partial order $\preceq$ defined on Y by a polyhedral convex cone both the $\preceq$-epigraph and the $\preceq$-hypograph of P can be represented as the union of finitely many convex polyhedral subsets of $X \times Y$. When the space Y is ordered by a minihedral cone or equivalently when Y is a vector lattice the collection $\mathcal{F}(X,Y)$ endowed with standard pointwise algebraic operations and the pointwise ordering is a vector lattice too. In the paper we show that the collection of piecewise affine mappings coincides with the smallest vector sublattice of $\mathcal{F}(X,Y)$ containing all affine mappings. Moreover we prove that each convex (with respect to an ordering of Y by a minihedral cone) piecewise affine mapping is the least upper bound of finitely many affine mappings. The collection of all convex piecewise affine mappings is a convex cone in $\mathcal{F}(X,Y)$ the linear envelope of which coincides with the vector subspace of all piecewise affine mappings.

MSC 2010: Primary: 52A07 Secondary: 26A27

Key words: polyhedral sets, piecewise affine mappings, minimax representation.

1. Introduction

Roughly speaking, a mapping is piecewise affine if it is "glued" with finitely many "pieces" of affine mappings (for an exact definition see Section 3). By simplicity, this class of nonlinear mappings is the most close one to linear and affine mappings. Along with this, every continuous nonlinear mapping can be approximated on any compact set by piecewise affine mappings with an arbitrary accuracy. Due to such good approximating properties piecewise affine mappings are widely used in nonlinear analysis both in pure theoretical studies and in various applications (see, for instance, [1 − 9]). In the present paper we review the results of a number of articles [10 − 15] written by the author in the recent past and devoted to piecewise affine functions and mappings acting between finite-dimensional vector spaces. These articles were published mainly in not readily available issues in Russian. The main purpose of the paper is to make these results more available to the interested readers. In Section 2 we present some preliminary facts on polyhedral (not necessarily convex) sets. Various definitions of piecewise affine mappings and some of their properties are discussed in Section 3. Geometrical characteristic properties of piecewise affine mappings are presented in Section 4. In particular, it is proved that a mapping $P : X \to Y$ is piecewise affine if and only if for any partial order $\preceq$ defined on Y by a polyhedral convex cone both the $\preceq$-epigraph and the $\preceq$-hypograph of P can be represented as the union of finitely many convex polyhedral subsets of $X \times Y$. In Section 6 for the case when Y is a vector lattice a number of analytical representations of piecewise affine mappings are presented. The main above results are specified in Section 7 to piecewise linear mappings.

2. Preliminaries on polyhedral (not necessarily convex) sets

Let X be a finite-dimensional normed space and let X^* be its dual space whose elements are linear functions on X.

A hyperplane in X is the set $H(a^*,\alpha) := \{x \in X \mid a^*(x) = \alpha\}$, where $a^* \in X^*$, $a^* \neq 0$, $\alpha \in \mathbb{R}$. Each hyperplane generates in X two closed halfspaces

$$H_{\leq}(a^*,\alpha) := \{x \in X \mid a^*(x) \leq \alpha\} \text{ and } H_{\geq}(a^*,\alpha) := \{x \in X \mid a^*(x) \geq \alpha\}$$

and two complementary open halfspaces

$$H_{>}(a^*,\alpha) := \{x \in X \mid a^*(x) > \alpha\} \text{ and } H_{<}(a^*,\alpha) := \{x \in X \mid a^*(x) < \alpha\}.$$

Since $H_{\leq}(a^*,\alpha) = H_{\geq}(-a^*,-\alpha)$ and $H_{>}(a^*,\alpha) = H_{<}(-a^*,-\alpha)$, we shall mainly present closed halfspaces in the form $H_{\leq}(a^*,\alpha)$ and open halfspaces in the form $H_{>}(a^*,\alpha)$.

A convex subset Q of X is said to be *polyhedral* [16 – 19] if it is the intersection of finitely many closed halfspaces. In other words a convex set $Q \subset X$ is polyhedral if it can be presented in the form $Q = \bigcap\limits_{j=1}^{k} H_{\leq}(a_j^*,\alpha_j)$, where $a_j^* \in X^*$, $\alpha_j \in \mathbb{R}$, $j = 1,2,\dots,k$. By the convention the whole space X is a polyhedral convex set too.

Since every convex polyhedral set is in fact a solution set of some finite system of linear inequalities, the theory of convex polyhedral sets is substantially developed as a part of the theory of linear inequalities [20, 21] and as a part of the linear programming theory [22,23] which studies problems of minimizing linear functions on convex polyhedral sets.

An arbitrary (not necessarily convex) set $Q \subset X$ is called *polyhedral* [24] if it is the union of finitely many convex polyhedral sets. (In [7] such sets were called piecewise polyhedral.)

It follows immediately from the above definition that any polyhedral set Q can be represented in the form

$$Q = \bigcup_{i=1}^{m} \bigcap_{j=1}^{k(i)} H_{\leq}(b_{ij}^*,\beta_{ij}), \tag{2.1}$$

where $b_{ij}^* \in X^*$, $\beta_{ij} \in \mathbb{R}$, $j = 1,\dots,k(i)$; $i = 1,\dots,m$.

Let $\mathcal{M}(X)$ be the Boolean lattice of all subsets of X with set operations of union and intersection as lattice operations and let $M(X)$ be the collection of all polyhedral subsets of X. It was shown in [11, 12] that $M(X)$ is the smallest sublattice in $\mathcal{M}(X)$, which contains all closed halfspaces of the space X. Thus polyhedral sets of X are exactly those which can be constructed from a finite family of closed halfspaces as a result of finitely many operations of union and intersection. It is follows from characteristics of general sublattice (see [25, 26]), that every polyhedral set Q can be represented in the alternative form

$$Q = \bigcap_{j=1}^{k} \bigcup_{i=1}^{m(j)} H_{\leq}(c_{ij}^*,\gamma_{ij}), \tag{2.2}$$

where $c_{ij}^* \in X^*$, $\gamma_{ij} \in \mathbb{R}$, $i = 1,\dots,m(j)$; $j = 1,\dots,k$.

In general the collections of closed halfspaces $H_{\leq}(b_{ij}^*,\beta_{ij}), j = 1,\dots,k(i), i = 1,\dots,m,$ and $H_{\leq}(c_{ij}^*,\gamma_{ij}), i = 1,\dots,m(i), j = 1,\dots,k,$ which are used for the representation of the same set Q in (2.1) and (2.2) can be different.

It was shown in [14] that for every polyhedral set $Q \subset X$ there exists a finite two-index family of closed halfspaces $\{H_{\leq}(a_{ij}^*,\alpha_{ij}),\ i = 1,\dots,m; j = 1,\dots,k\}$ such that

$$P = \bigcup_{i=1}^{m} \bigcap_{j=1}^{k} H_{\leq}(a_{ij}^*,\alpha_{ij}) = \bigcap_{j=1}^{k} \bigcup_{i=1}^{m} H_{\leq}(a_{ij}^*,\alpha_{ij}).$$

3. The definition and elementary properties of piecewise affine mappings

Let X and Y be finite-dimensional normed spaces..

A finite family $\sigma = \{M_1, \dots, M_k\}$ of convex polyhedral subsets $M_1, \dots, M_k$ of X is called *a polyhedral covering* of a polyhedral set $Q \subset X$, if

$$M_i \subset Q,\ i = 1, \dots, k;\ \text{and } Q = \bigcup_{i=1}^{k} M_i. \tag{3.1}$$

A family $\sigma = \{M_1, \dots, M_k\}$ is called *a polyhedral partition* of a polyhedral set $Q \subset X$, if it is a polyhedral covering and, in addition, the conditions $\mathrm{ri} M_i \cap \mathrm{ri} M_j = \varnothing$, $i, j = 1, \dots, k$, $i \neq j$, hold.

Here $\mathrm{ri} M$ stands for the relative interior of a convex set M.

A polyhedral covering (a polyhedral partition) $\sigma = \{M_1, \dots, M_k\}$ of the set Q is called *solid*, if each M_i, $i = 1, \dots, k$ has nonempty interior or, equivalently, if

$$\dim \mathrm{aff} M_i = \dim X,\ i = 1, \dots, k,$$

where $\mathrm{aff} M$ is the affine hull of the set M.

Definition 3.1 [13, 15]. A mapping $P : X \to Y$ is said to be *piecewise affine*, if there exists a polyhedral covering $\sigma = \{M_1, \dots, M_k\}$ of the vector space X and the collection of affine mappings $A_i : X \to Y, i = 1, \dots, k$, such that

$$P(x) = A_i(x),\ x \in M_i,\ i = 1, \dots, k.$$

In what follows we shall denote the collection of all piecewise affine mappings from X into Y by the symbol $PA(X, Y)$. When $Y = \mathbb{R}$ the collection $PA(X, \mathbb{R})$ of all piecewise affine functions from X into $\mathbb{R}$ will be denoted, for short, by $PA(X)$.

Example 3.1. Every affine and, consequently, linear mapping is piecewise affine. Hence, the following inclusions hold:

$$L(X, Y) \subset A(X, Y) \subset PA(X, Y).$$

where $L(X, Y)$ and $A(X, Y)$ are, respectively, the space of linear mappings and the space of affine ones from X into Y.

Example 3.2. The mappings $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $G : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, defined by $F(x_1, x_2) = \max\{x_1, x_2\}$ and $G(x_1, x_2) = \min\{x_1, x_2\}$, are piecewise affine. The polyhedral covering of X which is associated with F and G is $\{M_1, M_2\}$, where $M_1 := \{(x_1, x_2) \in \mathbb{R} \times \mathbb{R} \mid x_1 \leq x_2\}$, $M_2 := \{(x_1, x_2) \in \mathbb{R} \times \mathbb{R} \mid x_1 \geq x_2\}$.

Example 3.3. The mapping $G_\lambda : X \ni x \to \lambda x \in X$, where λ is an arbitrary fixed real number, and the mapping $F : X \times X \ni (x_1, x_2) \to x_1 + x_2 \in X$ are linear and, consequently, piecewise affine.

The above definition of piecewise affine mappings differs from the commonly accepted one (see, for instance, [4, 5]) in the requirement that a family σ is a covering of X rather than a solid partition of X. This requirement is less restrictive and therefore it is more convenient in proving some properties of piecewise affine mappings. Moreover, the above definition is actually equivalent to the common one. For the first time, this fact was proved in [13]. Later, another proof was given in [7]. Below, we will reproduce the proof given in [13].

Proposition 3.1. *If $\sigma = \{M_1, \ldots, M_k\}$ is a polyhedral covering of a convex polyhedral set Q with* $\mathrm{int}Q \neq \varnothing$ *then the family*

$$\sigma' = \{M_i \in \sigma \mid \mathrm{int}M_i \neq \varnothing\}$$

is a solid polyhedral covering of the set Q.

Proof. First we prove the proposition for the case when $Q = X$. Let $\hat{\sigma} := \{M_i \in \sigma \mid \mathrm{int}M_i = \varnothing\}$. The set $\hat{M} := \cup\{M \mid M \in \widehat{\sigma}\}$ is closed and nowhere dense in X (as the union of finitely many closed and nowhere dense sets [27, p. 114, Proposition 1]), and, hence, its complement $X\backslash\hat{M}$ is everywhere dense in X. Then, since $X\backslash\hat{M} \subset M'$, the set $M' := \cup\{M \mid M \in \sigma'\}$ is dense in X too. Because M' is closed we get $M' = X$. This proves the assertion of the proposition for the case $Q = X$.

Now we suppose that $\sigma = \{M_1, \ldots, M_k\}$ is a polyhedral covering of an arbitrary convex polyhedral set Q with $\mathrm{int}Q$ and $\sigma' = \{M \in \sigma \mid \mathrm{int}M \neq \varnothing\}$. We can present the set Q in the form $Q = \bigcap\limits_{i=1}^{p} H_{\le}(a_i^*, \alpha_i)$, where $a_i^* \in X^*\backslash\{0\}$, $\alpha_i \in \mathbb{R}$, $i = 1, \ldots, p.$. It is not difficult to see, that the family $\sigma \cup \{H_{\ge}(a_1^*, \alpha_1), \ldots, H_{\ge}(a_p^*, \alpha_p)\}$ is a polyhedral covering of the whole space X. Then, as it has been proved in the first part of this proof, the family $\sigma' \cup \{H_{\ge}(a_1^*, \alpha_1), \ldots, H_{\ge}(a_p^*, \alpha_p)\}$ is a solid polyhedral covering of X. Now we conclude from the equality $\mathrm{int}Q = \bigcap\limits_{i=1}^{p} H_{<}(a_i^*, \alpha_i)$ (see [16, Theorem 6.5]) that $\mathrm{int}Q \subset \cup\{M \mid M \in \sigma'\} \subset Q$ and, since the set Q is convex and closed, we have $\mathrm{cl}(\mathrm{int}Q) = Q$ (see., for instance, [16, Theorem 6.3]) and, consequently, $Q \subset \mathrm{cl}(\cup\{M \mid M \in \sigma'\}) = \cup\{M \mid M \in \sigma'\} \subset Q$.

It is easy to see that when a family σ is a polyhedral partition of a set Q, then σ' is a solid polyhedral partition of Q.

This completes the proof of Proposition 3.1.

Proposition 3.2. *For any polyhedral covering $\sigma = \{M_1, \ldots, M_k\}$ of a polyhedral set Q there exists a polyhedral partition $\omega = \{D_1, \ldots, D_m\}$ of the set Q such that every $D_j \in \omega$ is contained in some $M_i \in \sigma$.*

First of all we prove the following auxiliary lemma.

Lemma 3.1. *Let $\{H(a_i^*, \alpha_i),\ i \in S\}$ be a family of hyperplanes in X indexed by elements of a set S and let the set*

$$D_I = (\bigcap_{i \in I} H_{\le}(a_i^*, \alpha_i)) \bigcap (\bigcap_{i \in S\backslash I} H_{\ge}(a_i^*, \alpha_i))$$

be associated with every (possibly, empty) subset I of S.

Then for any subsets I, $J \subset S$ one has either $D_I = D_J$ or $\mathrm{ri}D_I \cap \mathrm{ri}D_J = \varnothing$.

Proof of Lemma 3.1. Notice that for every $i \in K := (I\backslash J)\bigcup(J\backslash I)$ the sets D_I and D_J lies in different closed halfspaces generated by the hyperplane $H(a_i^*, \alpha_i)$.

When there is $i \in K$ such that at least one of the sets D_I or D_J does not lie whole in $H(a_i^*, \alpha_i)$, the convex sets D_I and D_J are properly separated by the hyperplane $H(a_i^*, \alpha_i)$ and, hence, $\mathrm{ri}D_I \bigcap \mathrm{ri}D_J = \varnothing$ (see. [16, Theorem 11.3]).

In the case when D_I, $D_J \subset H(a_i^*, \alpha_i)$, for all $i \in K$, we have

$$D_I = (\bigcap_{i \in K} H(a_i^*, \alpha_i)) \bigcap (\bigcap_{i \in I\backslash K} H_{\le}(a_i^*, \alpha_i)) \bigcup (\bigcup_{i \in S\backslash(I \cup K)} H_{\ge}(a_i^*, \alpha_i)),$$

$$D_J = (\bigcap_{i \in K} H(a_i^*, \alpha_i)) \bigcap (\bigcap_{i \in J\backslash K} H_{\le}(a_i^*, \alpha_i)) \bigcup (\bigcup_{i \in S\backslash(J \cup K)} H_{\ge}(a_i^*, \alpha_i)).$$

Since $I\backslash K = J\backslash K = I\cap J$ and $I\cup K = J\cup K = I\cup J$, then $D_I = D_J$. This proves the lemma.

Proof of Proposition 3.2. Let $\sigma = \{M_1, \ldots, M_k\}$ be a polyhedral covering of the set Q. Every set $M_i, i = 1, \ldots, k$ can be presented in the form $M_i = \bigcap\limits_{j\in S(i)} H_{\le}(a_j^*, \alpha_j)$, where $S(i)$ is a finite family of indices. Let $\{H_{\le}(a_j^*, \alpha_j),\ j\in S := \bigcup\limits_{i=1}^{m} S(i)\}$ be the collection of all hyperplanes which are used in the above presentations of the sets $\{M_1, \ldots, M_k\}$. With every nonempty subset $I\subset S$ we associate the set $D_I = (\bigcap\limits_{j\in I} H_{\le}(a_j^*, \alpha_j)) \bigcap\ \bigcap(\bigcap\limits_{j\in S\backslash I} H_{\ge}(a_j^*, \alpha_j))$. Notice that for some $I,\ J\subset S,\ I\ne J,$ we can have $D_I = D_J$. Let us consider the family ω' consisting of all nonempty subsets D_I, that lie in one of subsets $M_i,\ i = 1, \ldots, k$. It is evident that $\cup\{D \mid D\in\omega'\}\subset Q$.

Let x be a point of Q and let $i\in\{1, 2, \ldots, m\}$ be such that $x\in M_i$. It is easy to see that $x\in D_J$, where $J = \{j\in S \mid x\in H_{\le}(a_j^*, \alpha_j)\}$. Moreover, since $S(i)\subset J$, then $D_J\subset M_i$ and, hence, $D_J\in\omega'$. As x is an arbitrary point of Q, we get $Q\subset\bigcup\{D \mid D\in\omega'\}$. Consequently, the family ω' is a covering of Q. Besides it follows from Lemma 1 that different subsets of ω' do not intersect each other by their relative interiors. Thus, the subfamily ω of nonempty different subsets of ω' is in fact a polyhedral partition of Q and, moreover, each subset of ω is contained in some subset of the covering σ. This completes the proof of Proposition 3.2.

The next theorem follows immediately from Proposition 3.1 and Proposition 3.2.

Theorem 3.1. *A mapping $P : X\to Y$ is piecewise affine if and only if there exists a solid polyhedral partition σ' of the space X such that P coincides with some affine mapping on each subset of σ'.*

Theorem 3.1 proves that Definition 3.1 of piecewise affine mappings is in fact equivalent to the conventional one.

We complete this section with describing some properties of the space of piecewise affine functions.

Theorem 3.2. *The composition of piecewise affine mappings is a piecewise affine mapping.*

Proof. Let X, Y and Z be finite dimensional normed spaces and let $P\in PA(X, Y)$ and $Q\in PA(Y, Z)$ be piecewise affine mappings. Let $\sigma = \{M_1, \ldots, M_k\}$ and $\delta = \{D_1, \ldots, D_p\}$ be polyhedral covering of X and Y which correspond, respectively, to P and Q; $\{P_1, \ldots, P_k\}$ and $\{Q_1, \ldots, Q_p\}$ the collections of affine mappings from $A(X, Y)$ and from $A(Y, Z)$ such that $P(x) = P_i(x),\ x\in M_i,\ i = 1, \ldots, k;\ Q(y) = Q_j(y),\ y\in D_j,\ j = 1, \ldots, p$. The sets $C_{ij} := M_i\cap P_i^{-1}(D_j),\ i = 1, \ldots, k,\ j = 1, \ldots, p,$ are convex and polyhedral (it follows from properties of convex polyhedral sets [13 – 16]) and form a polyhedral covering of X. Besides, on each set C_{ij} the composition $Q\circ P$ coincides with the affine mapping $Q_j\circ P_i$. This proves the theorem.

Theorem 3.2 implies a number of important corollaries.

Corollary 3.1. *The collection $PA(X, Y)$ endowed with standard pointwise operations of addition and multiplication by reals is a vector space.*

The validity of this assertion immediately follows from Theorem 3.2 and Example 3.3.

Let $\dim Y = n$ and let $\{e_1, \ldots, e_n\}$ be a vector basis of Y. As is well known any mapping $P : X\to Y$ can be uniquely associated with its *coordinate functions* $p_i : X\to\mathbb{R},\ \ i = 1, \ldots, n$, such that $P(x) = p_1(x)e_1 + \ldots + p_n(x)e_n$ for all $x\in X$.

Corollary 3.2. *The mapping $P : X\to Y$ is piecewise affine if and only if its coordinate functions $p_i : X\to\mathbb{R},\ i = 1, \ldots, n$, are piecewise affine.*

Proof. Since the mappings $\phi_i : \lambda \to \lambda e_i,\ \ i = 1,\dots,n,$ from $\mathbb{R}$ into Y are linear, the sufficient part of Corollary 3.2 follows directly from Theorem 3.2 and Corollary 3.1.

To prove the necessary part we choose in Y^* the basis $\{e_1^*,\dots,e_n^*\}$ which is dual to the basis $\{e_1,\dots,e_n\}$ of Y. It means that $\langle e_i^*, e_i\rangle = 1$ for all $i = 1,\dots,n$ and $\langle e_i^*, e_j\rangle = 0$ for $i,j = 1,\dots,n;\ i \neq j$. Since $e_i^*,\ i = 1,\dots,n,$ are linear functions on $Y,$ it follows from Theorem 3.2 that the functions $p_i(x) = e_i^*(P(x)),\ i = 1,\dots,n,$ are piecewise affine.

4. Geometrical characteristic properties of piecewise affine mappings

Directly from the definition of piecewise affine mappings we can see that the graph of a piecewise affine mapping is a union of finitely many convex polyhedral subsets, each of which is a part of the graph of an affine mapping, and, consequently, the graph of a piecewise affine mapping is a polyhedral (but, in general, nonconvex) set. In spite of apparent evidence of this observation we provide it with a proof.

Theorem 4.1. *A mapping* $P : X \to Y$ *is piecewise affine if and only if its graph* $\mathrm{graph}P := \{(x,y) \in X \times Y \mid P(x) = y\}$ *is a polyhedral set in* $X \times Y$ *.*

Proof. *Necessity.* Let $P : X \to Y$ be a piecewise affine mapping and let $\sigma = \{M_1,\dots,M_k\}$ and $\mathcal{A} = \{A_1,\dots,A_k\}$ be the polyhedral partition of the space X and the collection of affine functions associated with P. Then the equality

$$\mathrm{graph}P = \bigcup_{i=1}^{k} \Big((M_i \times Y) \bigcap \mathrm{graph}A_i \Big),$$

holds and we see that the graph of P is a polyhedral set in $X \times Y$.

Sufficiency. First we consider the case when $Y = \mathbb{R}$. Let $\mathrm{graph}P = \bigcup_{i=1}^{k} G_i$, where $\Sigma = \{G_1,\dots,G_k\}$ is a family of convex polyhedral sets in $X \times \mathbb{R}$. Then the family $\sigma := \{M_1,\dots,M_k\}$ with $M_i := \mathrm{pr}_X G_i$ ($\mathrm{pr}_X G_i$ stands for the projection of G_i on X) is a polyhedral covering of X. In view of Proposition 3.1 the subcollection $\sigma' = \{M_i \in \sigma \mid \mathrm{int}M_i \neq \varnothing\}$ is a solid polyhedral covering of X. Let M_i be a subset of σ' and G_i be a subset of Σ corresponding to M_i. Suppose that $\dim X = m$. Since $\mathrm{int}M_i \neq \varnothing$ we can choose in M_i a collection $\{x_0,\dots,x_m\} \subset M_i$ of affinely independent points [16]. It is not difficult to verify that the collection $\{(x_0,P(x_0)),\dots,(x_m,P(x_m))\}$ of points of G_i is also affinely independent and, consequently, $\dim \mathrm{aff}\, G_i \geq m$. But the dimension of the affine hull of G_i can not be equal to $m+1$ because it contradicts the single-valuedness of P . Therefore, $\dim \mathrm{aff}\, G_i = m$ and, hence, the affine hull $\mathrm{aff}\, G_i$ is a hyperplane in $X \times \mathbb{R}$. Then there are $a_i^* \in X^*$, and $\alpha_i, \beta_i \in \mathbb{R}$ such that $\mathrm{aff}\, G_i = \{(x,\xi) \in X \times \mathbb{R} \mid a_i^*(x) + \alpha_i \xi = \beta_i\}$. Note that the coefficient α_i is not equal to zero, otherwise we had the inclusion $M_i \subset \{x \in X \mid a_i^*(x) = \beta_i\}$ that contradicts the condition $\mathrm{int}M_i \neq \varnothing$. Hence the hyperplane $\mathrm{aff}\, G_i$ is the graph of the affine function $h_i : x \to (a_i^*(x) - \beta_i)/\alpha_i$. Thus the solid polyhedral covering $\sigma' = \{M_i \in \sigma \mid \mathrm{int}M_i \neq \varnothing\}$ of the space X is associated with the collection of affine functions $h_i : X \to \mathbb{R}$ such that $P(x) = h_i(x)$ for all $x \in M_i$. This proves that the function $P : X \to \mathbb{R}$ is piecewise affine.

Suppose now that $\dim Y = n > 1$ and choose in Y a vector basis $\{e_1,\dots,e_n\}$. Let $p_i : X \to \mathbb{R},\ i = 1,\dots,n,$ be coordinate functions of the mapping P corresponding to this basis. Then $\mathrm{graph}p_i = r_i(\mathrm{graph}P),\ i = 1,\dots,n$, where the mapping $r_i : X \times Y \to X \times \mathbb{R}$ is defined by the equality $r_i(x,y) = (x, \langle y, e_i^*\rangle)$. Here $\{e_1^*,\dots,e_n^*\}$ stand for the basis in Y^* which is dual to the basis $\{e_1,\dots,e_n\}$. Since the mappings $r_i,\ i = 1,\dots,n,$ are linear and $\mathrm{graph}P$ is polyhedral, $\mathrm{graph}p_i$ is polyhedral too. Consequently, as it was proved above, the coordinate functions

p_i, $i = 1, \ldots, n$, are piecewise affine. Then it follows from Corollary 3.2 that the mapping P is piecewise affine too. This completes the proof of the theorem.

A partial order $\preceq$ defined on Y will be called *polyhedral* if it is compatible with algebraic operations on Y and its positive cone $Y^+_{\preceq} := \{y \in Y \mid 0 \preceq y\}$ is polyhedral.

Theorem 4.2. *A mapping $P : X \to Y$ is piecewise affine if and only if for any polyhedral partial order $\preceq$ defined on Y both the $\preceq$-epigraph* $\mathrm{epi}_{\preceq} P := \{(x, y) \in X \times Y \mid P(x) \preceq y\}$ *and the $\preceq$-hypograph* $\mathrm{hyp}_{\preceq} P := \{(x, y) \in X \times Y \mid y \preceq P(x)\}$ *of P are polyhedral sets in $X \times Y$.*

Proof. Since for any polyhedral partial order $\preceq$ defined on Y both $\preceq$-epigraph $\mathrm{epi}_{\preceq} A$ and $\preceq$-hypograph $\mathrm{hyp}_{\preceq} A$ of any affine mapping $A : X \to Y$ are convex polyhedral sets in $X \times Y$ (it follows, for instance, from [16, Theorem 3.6]), the necessary part of the theorem follows from the equalities

$$\mathrm{epi}_{\preceq} P = \{\bigcup_{i=1}^{k}((M_i \times Y) \bigcap \mathrm{epi}_{\preceq} A_i)\},$$

$$\mathrm{hyp}_{\preceq} P = \{\bigcup_{i=1}^{k}((M_i \times Y) \bigcap \mathrm{hyp}_{\preceq} A_i)\},$$

where $\{M_1, \ldots, M_k\}$ is a polyhedral covering of X, and $\{A_1, \ldots, A_k\}$ is a collection of affine functions such that $p(x) = A_i(x)$ for all $x \in M_i$, $i = 1, \ldots, k$.

The sufficient part follows from the equality $\mathrm{graph} P = \mathrm{epi}_{\preceq} P \cap \mathrm{hyp}_{\preceq} P$ and Theorem 4.1.

Corollary 4.1 [10]. *A real-valued function $p : X \to \mathbb{R}$ is piecewise affine if and only if its epigraph* $\mathrm{epi} p := \{(x, \alpha) \in X \times \mathbb{R} \mid p(x) \leq \alpha\}$ *and its hypograph* $\mathrm{hyp} p := \{(x, \alpha) \in X \times \mathbb{R} \mid p(x) \geq \alpha\}$ *are polyhedral sets.*

5. Analytical characterizations of piecewise affine mappings

In what follows we shall suppose that the vector space Y is endowed with a partial order $\preceq$ with respect to which Y is an Archimedean vector lattice [25, 26]. As it is known (see, for instance, [28]), a finite-dimensional ordered vector space $(Y, \preceq)$ is an Archimedean vector lattice if and only if its positive cone $Y^+_{\preceq} := \{y \in Y \mid 0 \preceq y\}$ is a conic convex hull of some vector basis $\{e_1, e_2, \ldots, e_n\}$ of Y, where $n = \mathrm{dim} Y$. Thus, the assumption that Y is an Archimedean vector lattice does not restrict the generality of consideration because it is equivalent to the choice of some vector basis in Y. Below without additional mentions we shall assume that a vector bases $\{e_1, \ldots, e_n\}$ of Y and a partial order $\preceq$ defined on Y are compatible in such way that $Y^+_{\preceq} := \mathrm{coconv}\{e_1, \ldots, e_n\}$, where $\mathrm{coconv} M$ denotes a conic convex hull of a set M.

Under above assumptions every mapping $F : X \to Y$ can be represented as $F(x) = f_1(x)e_1 + f_2(x)e_2 + \ldots + f_n(x)e_n$, where $f_i : X \to Y$, $i = 1, 2, \ldots, n$, are real-valued functions called the coordinate functions of F.

The collection $\mathcal{F}(X, Y)$ of all mappings from X into Y, endowed with standard pointwise algebraic operations and with the partial order $\preccurlyeq$ defined by

$$F \preccurlyeq G \Longleftrightarrow F(x) \preceq G(x) \ \forall x \in X \Longleftrightarrow f_i(x) \leq g_i(x) \ \forall x \in X, \, i = 1, \ldots, n, \tag{5.1}$$

(here $\{f_1, \ldots, f_n\}$ and $\{g_1, \ldots, g_n\}$ stand for coordinate functions of F and G, respectively) is a vector lattice with

$$\sup\{F, G\}(x) = (\max\{f_1(x), g_1(x)\}, \ldots, \max\{f_n(x), g_n(x)\})$$

and

$$\inf(F,G)(x) = (\min\{f_1(x),\, g_1(x)\},\ldots,\min\{f_n(x),\, g_n(x)\})$$

as lattice operations.

It is not difficult to see that, whenever F and G are piecewise affine mappings, $\sup\{F,G\}$ and $\inf(F,G)$ are also piecewise affine ones. Thus the following assertion is true.

Proposition 5.1. *The collection of all piecewise affine mappings $PA(X,Y)$ from X into Y with pointwise algebraic operations and with the partial order $\preccurlyeq$ defined by* (5.1) *is a vector sublattice of the vector lattice $\mathcal{F}(X,\, Y)$.*

A mapping $F : X \to Y$ is said to be $\preceq -$ *convex* (convex with respect to a partial order $\preceq$) if

$$F(\alpha x + \beta y) \preceq \alpha F(x) + \beta F(y),$$

for all $x, y \in X$ and all $\alpha, \beta \geq 0,\ \alpha + \beta = 1$.

Proposition 5.2. *A piecewise affine mapping $P : X \to Y$ is $\preceq -$ convex if and only if it can be represented in the form*

$$P(x) = \sup_{j\in J} A_j(x),\ x \in X, \tag{5.2}$$

where $\{A_j : X \to Y,\ j \in J\}$ is a finite family of affine mappings from X into Y and $\sup$ *is the least upper bound in the vector lattice Y.*

Proof. Since P is $\preceq$-convex and piecewise affine, the coordinate functions $p_i : X \to Y, i = 1,\ldots,n,$ of P are convex and piecewise affine. Due to Proposition 3.1 of [7] every coordinate function $p_i : X \to Y\,(i = 1,\ldots,n)$, can be represented in the form $p_i(x) = \max\limits_{j\in J_i}(a_j^*(x) + \alpha_j),\ x \in X,$ $i = 1,\ldots,k,$ where J_i are finite family of indices, $a_j^* : X \to \mathbb{R},\ j \in J_1 \cup J_2 \cup \ldots \cup J_n,$ are real-valued linear functions. Define the set of multi-indices $J = J_1 \times J_2 \times \ldots \times J_n$ and associate with every $j = (j_1,\ldots,j_n) \in J$ the affine mapping $A_j = (a_{j_1}^*(\cdot)+\alpha_{j_1},\ldots,a_{j_n}^*(\cdot)+\alpha_{j_n}) : X \to Y$. It is not difficult to verify that $P(x) = \sup\limits_{j\in J} A_j(x),\ x \in X,$ where sup is the least upper bound in the vector lattice Y.

The converse assertion follows from the inequality

$$\sup_{i\in J}(a_i + b_i) \leq \sup_{i\in J} a_i + \sup_{i\in J} b_i$$

where $\{a_i \mid i \in J\}$ and $\{b_i \mid i \in J\}$ are finite family of vectors in Y.

This completes the proof.

Theorem 5.1. *Let $P : X \to Y$ be a mapping from X into Y.*

The following assertions are equivalent:

a) $P : X \to Y$ is piecewise affine;

b) $P : X \to Y$ can be represented in the form

$$P(x) = \inf_{1\leq i\leq k}\ \sup_{1\leq j\leq m(i)} A_{ij}(x), \quad x \in X, \tag{5.3}$$

where $A_{ij} : X \to Y,\ j = 1,\ldots,m(i),\ i = 1,\ldots,k$, are piecewise affine mappings;

c) $P : X \to Y$ can be represented in the form

$$P(x) = \sup_{1\leq i\leq k}\ \inf_{1\leq j\leq m(i)} B_{ij}(x), \quad x \in X, \tag{5.4}$$

where $B_{ij} : X \to Y,\ j = 1, \ldots, m(i),\ i = 1, \ldots, k$ *, are piecewise affine mappings;*

d) $P : X \to Y$ *can be represented in the form*

$$P(x) = \sup_{1 \le i \le k} C_i(x) - \sup_{1 \le j \le m} D_j(x),\ x \in X, \tag{5.5}$$

where $C_i, D_j : X \to Y,\ i = 1, \ldots, k,\ j = 1, \ldots, m,$ *are piecewise affine mappings.*

Proof. The implications $b) \Rightarrow a)$, $c) \Rightarrow a)$ and $d) \Rightarrow a)$ immediately follow from the fact that the collection $PA(X, Y)$ of all piecewise affine mappings is a vector lattice (see Proposition 5.1) and that affine mappings belong to $PA(X, Y)$.

Let $P : X \to Y$ be a piecewise affine mapping. It follows from Corollary 3.2 that coordinate functions $p_s : X \to \mathbb{R},\ s = 1, \ldots, n$, corresponding to P also are peacewise affine. Due to Theorem 3.1 and Proposition 3.1 of [10] we can represent each function $p_s : X \to \mathbb{R},\ s = 1, \ldots, n$ in the form

$$p_s(x) = \min_{i \in I_s} q_i(x),\ x \in X,$$

where I_s is a finite family of indices and $q_i : X \to \mathbb{R},\ i \in I_s,$ are convex piecewise affine functions. For every $i = (i_1, \ldots, i_n) \in I := I_1 \times I_2 \times \ldots \times I_n$ we define the $\preceq$-convex piecewise affine mapping $P^{(i)} : X \to Y$, letting $P^{(i)}(x) = q_{i_1}(x)e_1 + q_{i_2}(x)e_2 + \ldots + q_{i_n}(x)e_n$. Then

$$P(x) = \inf_{i \in I} P^{(i)}(x),\ x \in X. \tag{5.6}$$

By Proposition 5.2 each $P^{(i)},\ i \in I,$ can be represented in the form (5.2) and, consequently, it follows from Equality (5.6), P can be represented in the form (5.3). It proves the implication $a) \Rightarrow b)$.

$b) \Rightarrow d)$ Suppose that a mapping $P : X \to Y$ is of the form (5.3) and let $G_i(x) = \sup\limits_{1 \le j \le m(i)} A_{ij}(x)$, $i = 1, \ldots, k$. Then

$$P(x) = \inf_{1 \le i \le k} G_i(x) = -\sup_{1 \le i \le k} \left(\sum_{s=1, s \ne i}^{k} G_s(x) - \sum_{s=1}^{k} G_s(x) \right) =$$

$$= \sum_{s=1}^{k} G_s(x) - \sup_{1 \le i \le k} \sum_{s=1, s \ne i}^{k} G_s(x)$$

Since a sum and a supremum of finitely many $\preceq$-convex piecewise mappings also are $\preceq$-convex piecewise affine mappings, both mappings $x \to \sum\limits_{s=1}^{k} G_s(x)$ and $x \to \sup\limits_{1 \le i \le k} \sum\limits_{s=1, s \ne i}^{k} G_s(x)$ are $\preceq$-convex and piecewise affine and, consequently, each of them can be represented in the form (5.2). It proves that the mapping P can be represented in the form (5.5).

$d) \Rightarrow c)$ Since P is of the form (5.5), letting $A_{ij}(x) = C_i(x) - D_j(x),\ \ x \in X,\ j = 1, \ldots, m,\ i = 1, \ldots, k$, we immediately get the representation (5.4). This completes the proof of the theorem.

It immediately follows from the assertion d) of Theorem 5.1 that for every piecewise affine mapping $P : X \to Y$ there exists finite two-index family of affine mappings $\{A_{ij} : X \to Y, i = 1, \ldots, m; j = 1, \ldots, k\}$ such that

$$P(x) = \inf_{1 \le i \le m} \sup_{1 \le j \le k} A_{ij}(x) = \sup_{1 \le j \le k} \inf_{1 \le i \le m} A_{ij}(x).$$

For piecewise affine functions (that is, for the case when $Y = \mathbb{R}$) a straightforward proof of the above representation was given in [14].

We complete this section with the theorem that characterizes approximative properties of piecewise affine mappings in the space of continuous mappings.

Theorem 5.2. *Let $\epsilon > 0$ be an arbitrary positive real. For any continuous mapping $F : X \to Y$ and for any compact subset Q of X there exists a piecewise affine mapping $P : X \to Y$ such that*

$$\max_{x \in Q} ||F(x) - P(x)||_Y < \epsilon.$$

The proof of this theorem follows from the lattice version of the Stone-Weirstrass theorem [29] (see, also, [30, Theorem 9.12]).

Corollary 5.1. *The space $P(X,Y)$ of piecewise affine mappings is dense in the space $C(X,Y)$ of continuous mappings from X into Y endowed with the local convex topology generated by the family of seminorm $||P||_Q = \max_{x \in Q} ||P(x)||$, where Q runs through compact sets in X.*

6. Piecewise linear mappings

In this section we present the main properties of piecewise linear mappings without proofs.

A piecewise affine mapping $P : X \to Y$ is called *piecewise linear*, if it is positively homogenous, that is, if $P(\lambda x) = \lambda P(x)$ for all $x \in X$ and $\lambda \geq 0$.

The collection of all piecewise linear mappings from X into Y will be denoted by $PL(X,Y)$. First of all we note that composition of piecewise linear mappings is a piecewise linear mapping too. If a basis is chosen and fixed in the space Y then a mapping $P : X \to Y$ is piecewise linear if and only if its coordinate functions are piecewise linear. A mapping $P : X \to Y$ is piecewise linear if and only if for any polyhedral partial order $\preceq$, defined on Y, $\preceq$-epigraph $\mathrm{epi}_{\preceq} P$ and $\preceq$-hypograph $\mathrm{hyp}_{\preceq} P$ are polyhedral cones in $X \times Y$.

If a partial order $\preceq$ is defined on Y by a minihedral convex cone or, equivalently, if Y ia an Archimedean vector lattice then with respect to standard pointwise algebraic operations and the partial order $\preccurlyeq$ defined by

$$P \preccurlyeq Q \Longleftrightarrow P(x) \preceq Q(x) \ \text{для всех } x \in X, \tag{6.1}$$

the collection $PL(X, Y)$ of piecewise linear mappings is a vector sublattice of $PA(X,Y)$ (and of $\mathcal{F}(X, Y)$ as well).

A piecewise linear mapping $P : X \to Y$ is $\preceq$-convex if and only if it can be represented in the form

$$P(x) = \sup_{j \in J} L_j(x), \ x \in X,$$

where $L_j : X \to Y, \ j \in J,$ are linear mappings from X into Y, J is a finite family of indices and the operation of supremum is a lattice operation in Y.

Theorem 6.1. *Let $P : X \to Y$ be a mapping from X into Y. The following assertions are equivalent:*
a) $P : X \to Y$ is piecewise linear;
b) $P : X \to Y$ can be presented in the form

$$P(x) = \inf_{1 \leq i \leq k} \sup_{1 \leq j \leq m(i)} L_{ij}(x), \quad x \in X,$$

where $L_{ij} : X \to Y, \ j = 1, \ldots, m(i), \ i = 1, \ldots, k$, are linear mappings;
c) $P : X \to Y$ can be presented in the form

$$P(x) = \sup_{1 \leq i \leq k} \inf_{1 \leq j \leq m(i)} L_{ij}(x), \quad x \in X,$$

where $L_{ij} : X \to Y,\ j = 1, \dots, m(i),\ i = 1, \dots, k$ *are linear mappings;*
d) $P : X \to Y$ *can be presented as difference of two* $\preceq$*-convex piecewise linear mappings, that is*

$$P(x) = \sup_{1 \le i \le k} U_i(x) - \sup_{1 \le j \le m} V_j(x),\ x \in X,$$

where $U_i, V_j : X \to Y,\ j = 1, \dots, m,\ i = 1, \dots, k$ *, are linear mappings.*

In assertions b), c) and d) operations of supremum and infimum are lattice operations in Y*.*

Let $\mathcal{P}(X, Y)$ be the vector space of positively homogenous continuous mappings from X into Y. The space $\mathcal{P}(X, Y)$ is a Banach one with respect to the norm

$$||P|| = \sup_{x \in S} ||P(x)||_Y,$$

where S is the unit sphere in X.

It follows from the lattice version of the Stone-Weierstrass theorem [29, 30]), that the vector subspace $PL(X, Y)$ of piecewise linear mappings is dense in $\mathcal{P}(X, Y)$.

Acknowledgment

The research was supported by the National Program of Fundamental Researches of Belarus, the grant ҮConvergence Ч 1.4.03Ұ.

References

[1] C.P. Rourke, B.J. Sanderson, Introduction in Piecewise-linear Topology, Springer, Berlin et al, 1972.

[2] A.I. Subbotin, Minimax inequalities and Hamilton-Jacobi equations, Nauka, Moscow, 1991 (in Russian).

[3] A.I. Subbotin, A piecewise-linear cost function for a differential game with simple moves, Trudy Mat. Inst. Steklov., 185 (1988), 242Џ251.

[4] D. Melzer, On the expressibility of piecewise-linear continuous functions as the difference of two piecewise-linear convex functions, Mathematical Programming Study 29(1986) 118 – 134.

[5] A. Kripfganz, R. Schulze, Piecewise affine functions as the difference of two convex functions, Optimization 18(1987) 23 – 29.

[6] S.G. Bartels, L. Kuntz, S. Scholtes, Continuous selections of linear functions and nonsmooth critical points, Nonlinear Analysis. Theory, Methods and Applications 24(1995) 385 – 407.

[7] R.T. Rockafellar, R.J.-B. Wets, Variational analysis, Springer-Verlag, Berlin, 1998.

[8] S. Ovchinnikov, Max-Min representation of piecewise linear functions, Contributions to Algebra and Geometry 43(2002) 297-302.

[9] Ch.D. Aliprantis, R. Tourky, Cones and Duality (Graduate Studies in Mathematics, V.84), Providence, RI, American Mathematical Society, 2007.

[10] V.V. Gorokhovik, O.I. Zorko, Polyhedral quasidifferentiability of real-valued functions, Doklady Akademii Nauk Belarusi 36(1992) 393 – 397 (in Russian).

[11] V.V. Gorokhovik, O.I. Zorko, Piecewise affine functions and polyhedrel sets, Optimization 31(1994) 3 – 17.

[12] V.V. Gorokhovik, O.I. Zorko, Nonconvex polyhedral sets and functions and their analytical representations, Doklady Akademii Nauk Belarusi 39(1995) 5 – 9 (in Russian).

[13] V.V. Gorokhovik, O.I. Zorko, Piecewise affine mappings, in: N.A. Evkhuta (Ed.), Operators and operator equations, Novocherkask State Technical University, Novocherkask, 1995, pp. 3 – 18 (in Russian).

[14] V.V. Gorokhovik, D.S. Malashevich, On analytical representations of polyhedral sets and piecewise affine functions, The National Academy of Sciences of Belarus, Proceedings of the Institute of Mathematics 5(2001) 15 – 19 (in Russian).

[15] V.V. Gorokhovik, Geometrical and analytical characterizations of piecewise affine mappings, The National Academy of Sciences of Belarus, Proceedings of the Institute of Mathematics 15 (2007) 22 – 32 (in Russian).

[16] B. Grünbaum, Convex Polytopes, John Wiley & Sons, London-New York-Sydney, 1967.

[17] R.T. Rockafellar, Convex analysis, Princeton University Press, Princeton, 1970.

[18] K. Leichtweiss, Konvexe Mengen, VEB Deitscher Verlag der Wissenschaften, Berlin, 1980.

[19] A. Brondsted, An Introduction to Convex Polytopes, Springer-Verlag, New York-Heidelberg-Berlin, 1983.

[20] H.W. Kuhn, A.W. Tucker (Eds.), Linear inequalities and related systems, Princeton University Press, Princeton, 1956.

[21] S.N. Chernikov, Linear inequalities, Nauka, Moscow, 1968 (in Russian).

[22] G.B. Dantzig, Linear programming and extensions, Princeton University Press, Princeton, 1963.

[23] S.A. Ashmanov, Linear programming, Nauka, Moscow, 1981(in Russian).

[24] S. Robinson, Some Continuity Properties of Polyhedral Multifunctions, Mathematical Programming Study, 14(1981) 206 – 214.

[25] G. Birkhoff, Lattice Theory, American Mathematical Society, Providence, R.I., 1967.

[26] G. Gratzer, General Lattice Theory, Springer, Berlin, 1978.

[27] N. Bourbaki Topologie generale. Ch. IX – X.– Moscow: Nauka, 1975 (Russian traslation). Использование вещественных чисел в общей топологии. Функциональные пространства. Сводка результатов. Словарь.

[28] I.M. Glazman, Yu.I. Lyubich, Finite-dimensional linear analysis: a systematic presentation in problem form, M.I.T. Press, Cambridge, Mass. 1974.

[29] N. Bourbaki Integration. Measures, integration of measures

[30] Ch.D. Aliprantis, K.C. Border, Infinite dimensional analysis, Berlin, Springer, 1999.